Problem Set 10 due November 25, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Represent the US flag as a 13×25 matrix A, where each entry represents a color as follows: the entry 1 represents red, the entry 0 represents white, and the entry -1 represents blue. Then write this matrix A as a sum of rank 1 matrices (i.e. akin to formula (239) in the lecture notes).

Note on vexillology: you may ignore the stars, so just assume that the top left corner is a full-blue 7×10 submatrix of A. The height of all the stripes is one row. *(20 points)*

Problem 2: If A is an $n \times n$ symmetric matrix with distinct eigenvalues $\lambda_1, ..., \lambda_n$ and orthonormal eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, what is the SVD of A? *(10 points)*

Problem 3: All matrices in this problem are 2×2. A lower/upper triangular matrix with 1's on the diagonal has one degree of freedom (the bottom-left/top-right entry); a diagonal matrix has two degrees of freedom (the diagonal entries). Hence the LDU factorization has $1 + 2 + 1$ degrees of freedom, which is precisely the number of degrees of freedom in choosing a 2×2 matrix.

(a) How many degrees of freedom does an orthogonal 2×2 matrix Q have? Explain. *(5 points)*

(b) What is the total number of degrees of freedom of the QR factorization? What about the total number of degrees of freedom of the SVD $U\Sigma V^T$? Explain. *(5 points)*

(c) What is the total number of degrees of freedom of $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal? Still in the 2×2 case. *(5 points)*

(d) Why didn't you get 4 in part (c)?
Hint: it's because matrices $Q\Lambda Q^T$ are special, i.e. they are ____________ *(5 points)*

Problem 4: Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}$.

(a) Compute the SVD of A, and the pseudo-inverse A^+. *(15 points)*

(b) Compute the vector $\boldsymbol{v}^+$ defined by formula (261) in the lecture notes, which will have the property that $A\boldsymbol{v}^+ = \boldsymbol{p}$ is as close to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as possible. *(5 points)*

(c) Compute all solutions to $A\boldsymbol{v} = \boldsymbol{p}$ and prove that $\boldsymbol{v}^+$ is the shortest one. *(10 points)*

Problem 5: (a) Compute the sixth roots of unity (i.e. the complex numbers z such that $z^6 = 1$) in both Cartesian (i.e. $a + bi$) and polar (i.e. $re^{i\theta}$) form. Draw them all on a picture of the plane. *(10 points)*

(b) Prove the double angle and triple angle formulas:

$$\cos(2\theta) = 2(\cos\theta)^2 - 1 \qquad \text{and} \qquad \cos(3\theta) = 4(\cos\theta)^3 - 3\cos\theta \tag{1}$$

by the following logic:

- think of $\cos\theta$ as the real part of the complex number $z = e^{i\theta} = a + bi$ where $a = \cos\theta$, $b = \sin\theta$
- then compute z^2 (respectively z^3) first in polar form, and
- finally compute z^2 (respectively z^3) in Cartesian form

By equating the results in the last two bullets, you should obtain (1). *(10 points)*